Design switching on graphs

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3 June 2025 🍰

Based on joint work with Ferdinand Ihringer (SUSTech)

Cospectral graphs

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$.

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Definition

Graphs with the same spectrum are **cospectral**.

Cospectral graphs

Conjecture (van Dam and Haemers, 2003)

Almost all graphs are determined by their spectrum.

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- Interesting for complexity theory

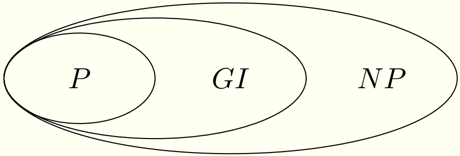

Figure: Is graph isomorphism an easy or hard problem?

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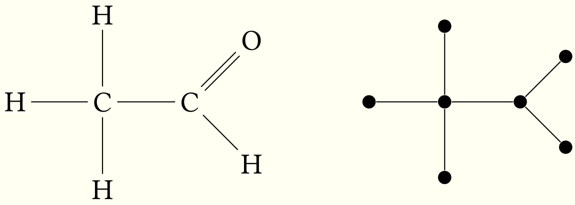

Figure: The molecular graph of acetaldehyde (ethanal).

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🙁 Almost all trees are **not** determined by their spectrum [Schwenk, 1973]

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🙂 Computational evidence [Brouwer and Spence, 2009]

n	3	4	5	6	7	8	9	10	11
ratio	1	1	0.941	0.936	0.895	0.861	0.814	0.787	0.789

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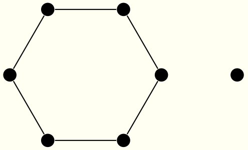
🙂 Exponentially many graphs are determined by their spectrum [Koval and Kwan, 2023]

How to find cospectral graphs

Theorem (Godsil and McKay, 1982)

Let Γ be a graph with a regular subgraph C of size 4 such that every vertex $x \notin C$ has $0, 2$ or 4 neighbours in C. For every $x \notin C$ that has exactly 2 neighbours in C, reverse its adjacencies with C. The resulting graph is cospectral with Γ.



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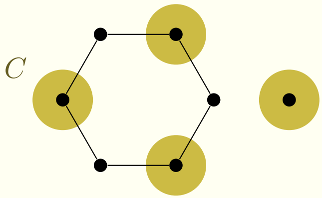

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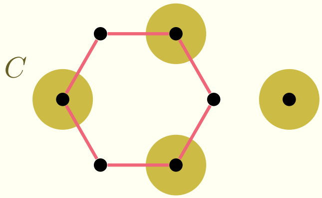

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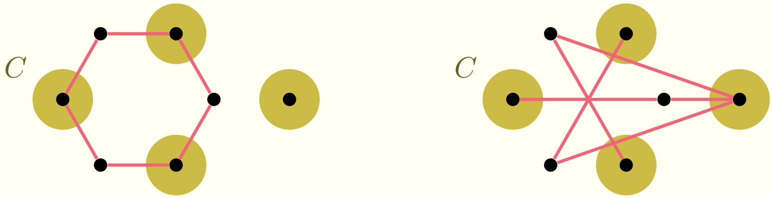

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Proof.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}.$$

□

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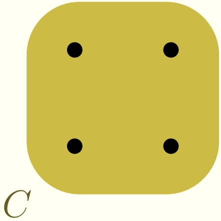

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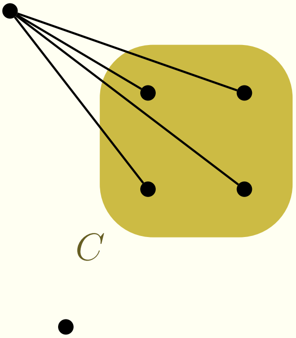

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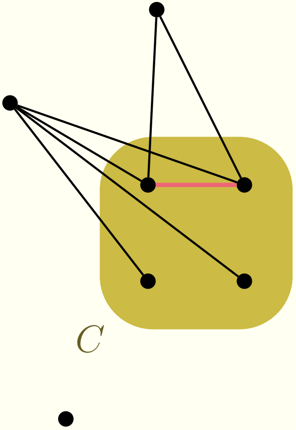

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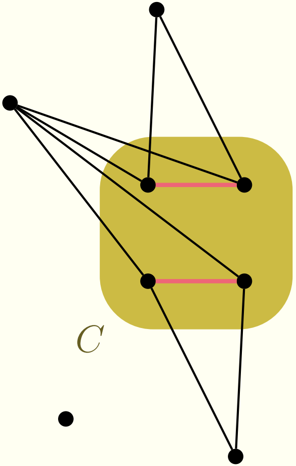

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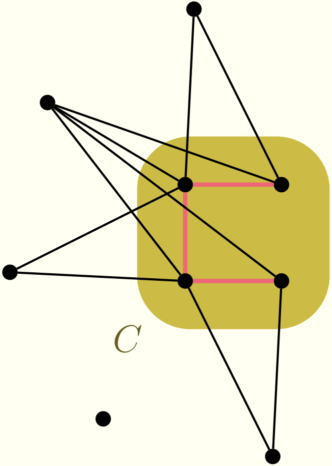

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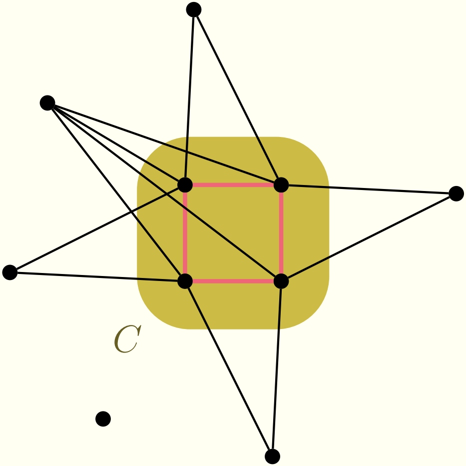

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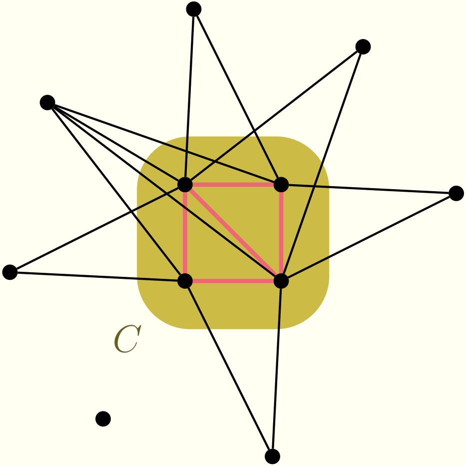

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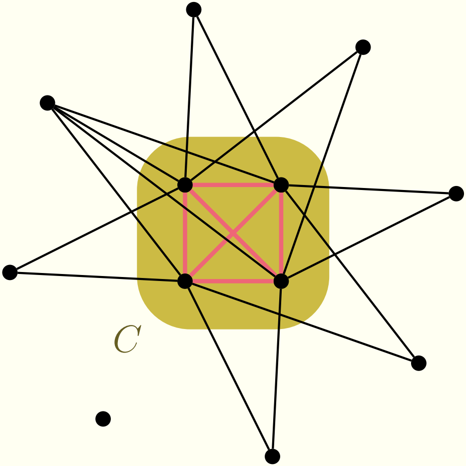

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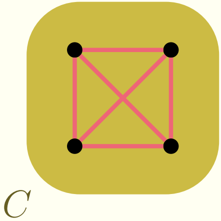

$\mathrm{AG}(2,2)$

How to find cospectral graphs

Definition

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a **switching set** with some conditions.

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"We here define **switching** and **switches** as certain local transformations that do not alter the basic parameters of a combinatorial structure." [Östergård, *Switching codes and designs*, 2012]

Table of contents

Switching methods

Definition

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a **switching set** with some conditions.

- GM-switching [Godsil and McKay, 1982]
- WQH-switching [Wang, Qiu and Hu, 2019]
- AH-switching [Abiad and Haemers, 2012]
  - Sun graph switching [Mao, Wang, Liu and Qiu, 2023]
  - Fano switching [Abiad, van de Berg and Simoens, 2025+]
  - Cube switching [Abiad, van de Berg and Simoens, 2025+]

Level 2: Abiad-Haemers switching

Theorem (Chan, Rodger and Seberry, 1986)

Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level 2 *and row sum* 1 *is one of the following:*

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\quad (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

$$(iii)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},\quad (iv)\ \frac{1}{2}\begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

where $I, J, O, Y = 2I - J$ *and* $Z = J - I$*, are* 2×2 *matrices.*

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Fano switching

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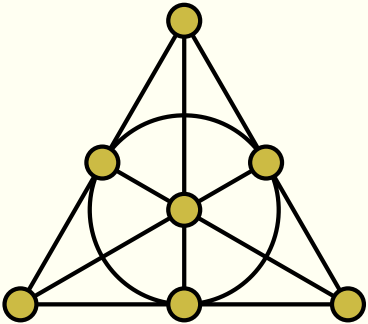
Fano switching

Abiad and Haemers (2012): algebraic conditions such that a conjugation of the adjacency matrix with $Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$, where

$$R = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

results in another adjacency matrix.

Fano switching



$\mathrm{PG}(2,2)$

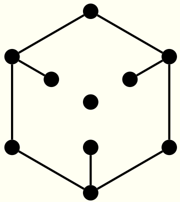
Fano switching

Theorem

Let Γ be a graph with a subgraph C whose vertices are identified as points of the Fano plane such that:

- *C is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in C.*
  - *If x has 3 neighbours in C, they form a line.*
  - *If x has 4 neighbours in C, they form the complement of a line.*

Let π be a permutation of the lines. For every $x \notin C$ that is (non)adjacent to the vertices of ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The resulting graph is cospectral with Γ.



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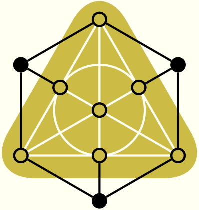

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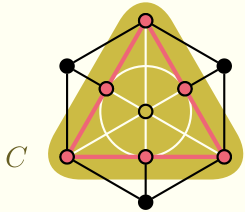

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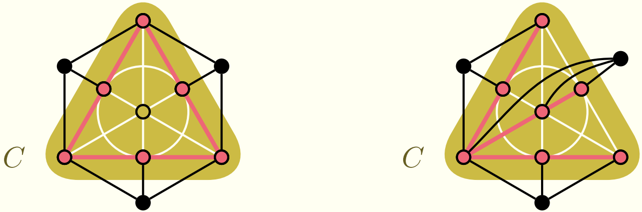

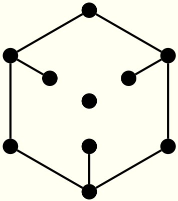

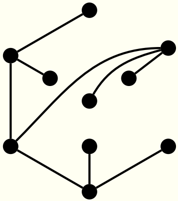

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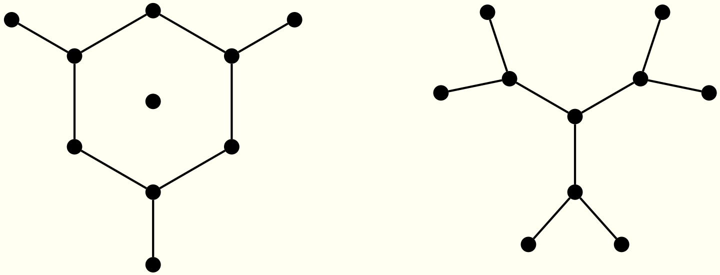

Both graphs have spectrum $\left\{(-\sqrt{5})^1, (-\sqrt{2})^2, (0)^3, (\sqrt{2})^2, (\sqrt{5})^1\right\}$.

Design switching

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Let π be a permutation of the blocks such that for all blocks B_i, B_j,

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)|\,.$$

For every $x \notin C$ adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with Γ.

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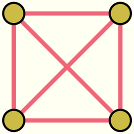

is an $(r = 3, \lambda = 1)$-design with incidence matrix

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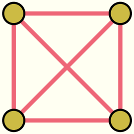

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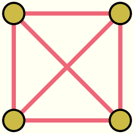
$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

Design switching



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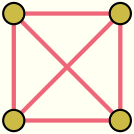

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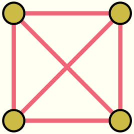
$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

Theorem (Godsil and McKay, 1982)

Let Γ be a graph with a regular subgraph C of size 4 such that every vertex $x \notin C$ has $0, 2$ or 4 neighbours in C. For every $x \notin C$ that has exactly 2 neighbours in C, reverse its adjacencies with C. The resulting graph is cospectral with Γ.

Design switching



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$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

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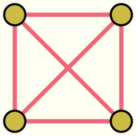

is an $(r = 4, \lambda = 2)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}. \end{array}$$

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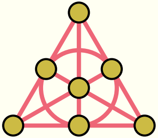
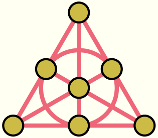


is an $(r = 3, \lambda = 1)$-design with incidence matrix

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
p_1	1	1	1	0	0	0	0
p_2	1	0	0	1	1	0	0
p_3	1	0	0	0	0	1	1
p_4	0	1	0	1	0	1	0
p_5	0	1	0	0	1	0	1
p_6	0	0	1	0	1	1	0
p_7	0	0	1	1	0	0	1

Any permutation of the lines π preserves pairwise intersection.

Design switching



is an $(r = 8, \lambda = 4)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

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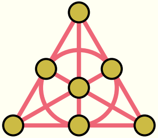
➤ Fano switching



is an $(r = 8, \lambda = 4)$-design with incidence matrix

$$
\begin{array}{c}
\\
p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7
\end{array}
\begin{array}{cccccccccccccccc}
 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & \overline{B_1} & \overline{B_2} & \overline{B_3} & \overline{B_4} & \overline{B_5} & \overline{B_6} & \overline{B_7} & \\
\end{array}
$$

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}.
$$

		B_1	B_2	B_3	B_4	B_5	B_6	B_7	$\overline{B_1}$	$\overline{B_2}$	$\overline{B_3}$	$\overline{B_4}$	$\overline{B_5}$	$\overline{B_6}$	$\overline{B_7}$	
p_1	0	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1
p_2	0	1	0	0	1	1	0	0	0	1	1	0	0	1	1	1
p_3	0	1	0	0	0	0	1	1	0	1	1	1	1	0	0	1
p_4	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1
p_5	0	0	1	0	0	1	0	1	1	0	1	1	0	1	0	1
p_6	0	0	0	1	0	1	1	0	1	1	0	1	0	0	1	1
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- Fano switching

Design switching

Theorem (Ihringer and Simoens, 2025+)

Let Γ be a graph with an edgeless or complete subgraph C whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block.
Let π be a permutation of the blocks such that for all blocks B_i, B_j,

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| \,.$$

For every $x \notin C$ adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with Γ.

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Proof. Define $R = \frac{1}{r-\lambda}\left(N(N^\pi)^T - \lambda J\right)$, where N^π is obtained from the incidence matrix N by permuting the columns with π.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} R & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R & O \\ O & I \end{pmatrix}.$$

□

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Theorem (Ihringer and Simoens, 2025+)

Let Γ be a graph with a subgraph C with adjacency matrix A_C such that $R^T A_C R$ is again an adjacency matrix whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block.
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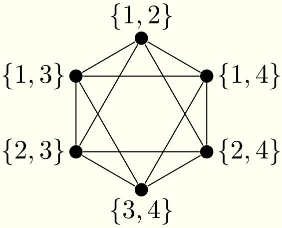

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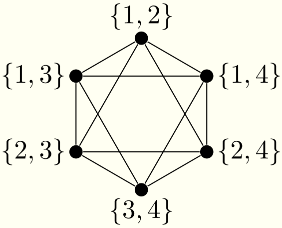

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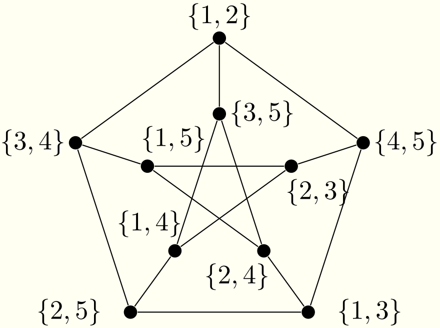

The Petersen graph $\overline{T_5}$.

Triangular graphs

Theorem (Chang and Hoffman, independently, 1959)

The triangular graph T_n is determined by its spectrum iff $n \neq 8$.

q-triangular graphs

Definition

The **q-triangular graph** $T_{q,n}$ has as vertices the 2-dimensional subspaces of $\mathbb{F}_q^n$ where two vertices are adjacent if they intersect.

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Proof. Consider the subgraph $T_{q,3}$ of all lines in a given plane $\mathrm{PG}(2, q) \subseteq \mathrm{PG}(n-1, q)$ and consider the design $D = (\mathcal{P}, \mathcal{B})$ where

$$\mathcal{P} = \{\text{lines of } \mathrm{PG}(2, q)\}$$
$$\mathcal{B} = \{\text{point pencils of } \mathrm{PG}(2, q)\}$$

Apply design switching, using any permutation π of $\mathcal{B}$ that is not an automorphism. This creates maximal cliques of size $q^2 + q$. □

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Let Γ_π denote the graph obtained from design switching $T_{n,q}$ with π. Then

$$\Gamma_{\pi_1} \cong \Gamma_{\pi_2}$$
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- Many strongly regular graphs with the same parameters.

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- All commonly known *indecomposable* switching methods can be reformulated as design switching.
- More general: π may also be a bijection between blocks of different designs.

Thank you for listening!